BUTLER GROUPS OF INFINITE RANK II

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ABSTRACT. A torsion-free abelian group G is called a Butler group if $\text{Bext}(G,T)$ $= 0$ for any torsion group T. We show that every Butler group G of cardinality $\aleph_1$ is a B_2-group; i.e., G is a union of a smooth ascending chain of pure subgroups G_α where $G_{\alpha+1} = G_\alpha + B_\alpha$, B_α a Butler group of finite rank. Assuming the validity of the continuum hypothesis (CH), we show that every Butler group of cardinality not exceeding $\aleph_\omega$ is a B_2-group. Moreover, we are able to prove that the derived functor $\text{Bext}^2(A, T) = 0$ for any torsion group T and any torsion-free A with $|A| \leq \aleph_\omega$. This implies that under CH all balanced subgroups of completely decomposable groups of cardinality $\leq \aleph_\omega$ are B_2-groups.

1. INTRODUCTION

All groups in this paper are (torsion-free) abelian groups. Undefined notations are standard as in [Fu]. A torsion-free abelian group is called a Butler group if $\text{Bext}(B, T) = 0$ for all torsion groups T. Here Bext is the subfunctor of Ext of all balanced-exact extensions. It is known [BS] that this definition coincides with the familiar one if B has finite rank, i.e., a pure subgroup of a completely decomposable group. The main result in this paper is that under the continuum hypothesis (CH) each Butler group B (of rank $\leq \aleph_\omega$) is a B_2-group [A]; i.e., B is the union of a smooth ascending chain of pure subgroups B_α, $\alpha < \lambda$ an ordinal and $B_{\alpha+1} = B_\alpha + L_\alpha$ for all $\alpha < \lambda$ where L_α is a Butler group of finite rank. This result partially answers questions raised in [BSS], [A], and elsewhere. We will, in fact, prove the following:

Theorem [CH]. *Let G be a torsion-free group of cardinality $\leq \aleph_\omega$.*

(α) *The following are equivalent:*

(1) $\text{Bext}(G, T) = 0$ *for all torsion groups T.*
(2) $\text{Bext}(G, T) = 0$ *for all Σ-cyclic torsion groups T.*
(3) *G is a B_2-group.*

(β) *The derived function $\text{Bext}^2(G, T) = 0$ for any torsion group T. (We have to use CH only if $|G| \geq \aleph_2$.)*

Part (α) of the preceding theorem extends results [A], [AH], [BS], [BSS], [D], and [DR1]. Part (β) was shown for $|G| \leq \aleph_1$ in [AH]. The statement that (3) implies (1) may be found in [BS]. Under $V = L$ it was shown in [DR2], with some help from [D], that (2) implies (3) in case $|G| \leq \aleph_1$.

As an immediate consequence of the theorem we obtain the following:

Corollary [CH]. *If G is a balanced subgroup of a completely decomposable group and $|G| < \aleph_\omega$, then G is a B_2-group.*

The proof of the theorem will show that if the corollary holds in general, we may drop the cardinality restriction in the theorem.

Albrecht and Hill [AH] showed (without CH) that the preceding corollary is true if $|G| \leq \aleph_1$.

In order to prove the above theorem we have to prove numerous auxiliary results, some of independent interest. These auxiliary results include the following: we show that any balanced subgroup of a completely decomposable group admits a $G(2^{\aleph_0})$-family (cf. [FH]) of balanced subgroups. The proof is a modification of one in [AH]. Using methods different from the proof of this result, we show that for *any* torsion-free group G of cardinality $\aleph_n$, $n \leq \omega$, there is a filtration $G = \bigcup_{\alpha < \aleph_n} G_\alpha$ into pure subgroups G_α such that $G_{\alpha+1}/G_\alpha$ has rank 1 and each G_α is separable in G in the sense of Hill; i.e., for all $g \in G$ there is a *countable* subset S of G_α such that for any $h \in G_\alpha$ there is some $s = s(h)$ in S for which $|g + h| \leq |g + s|$.

The main obstacle we have to overcome is the fact the pure subgroups of Butler groups (of infinite rank) are in general not again Butler groups. In order to show that balanced subgroups of Butler groups are Butler again, one needs to know that $\text{Bext}^2(G, T) = 0$ for torsion-free groups G. We were able to show this for all torsion-free groups G of cardinality not exceeding $\aleph_\omega$.

If T is a torsion group and G a torsion-free group, any pure subgroup $A \subseteq G$ gives rise to a natural map $\text{Bext}(G, T) \xrightarrow{i^*} \text{Bext}(A, T)$. Under the hypothesis that A is separable in G, $|G/A| \leq \aleph_1$, and each pure subgroup of finite rank of G is Butler, we are able to show that i^* is onto. This enables us to show that some balanced subgroups of completely decomposable groups are again Butler groups.

We will prove our theorem by cardinal induction. We slightly modify Hodges' version [H] of S. Shelah's singular-compactness theorem to fit our situation. The case of regular cardinals is then treated similarly to [EF, Lemma 9].

2. Preliminaries

Unless otherwise specified, all the groups that we consider here are additively written torsion-free abelian groups. We generally follow the notation and terminology of [Fu]. A height is a sequence $h = (h_p)$, indexed by the set P of primes p such that h_p is an ordinal or the symbol ∞ for each p. If G is an abelian group and $x \in G$, then $|x| = (h_p^G(x))$ is a height, where $h_p^G(x)$ is the

p-height of x in G. If S is a subset of a torsion-free group G, then $\langle S \rangle$, $\langle S \rangle_*$ denote, respectively, the subgroup and the pure subgroup generated by S. If $f : A \to B$ is a map and $S \subseteq A$, then $f \mid S$ denotes the restriction of f to S. If T is a torsion group, p is a prime, and $x \in T$, then T_p is the p-component of T and x_p the p-component of x in T.

A subgroup H of a torsion-free group G is said to be separable [HM] if to each $g \in G$ is a countable subset C of H such that to each $h \in H$, there is $s \in C$ such that $|g + h| \leq |g + s|$.

A torsion-free abelian group G is called a Butler group if $\mathrm{Bext}(G, T) = 0$ for all torsion groups T. A torsion-free group G is said to be a B_2-group if G is the union of a smooth chain $0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots$ of pure subgroups G_α such that, for each α, $G_{\alpha+1} = G_\alpha + B_\alpha$ where B_α is a finite-rank Butler group. In this case we call $G = \bigcup G_\alpha$ a B-filtration. Let H be a pure subgroup of a torsion-free group G. We say that H satisfies the torsion extension property (TEP, for short) in G or G has the TEP over H if, for any Σ-cyclic torsion group T, every homomorphism $f : H \to T$ extends to a homomorphism $g : G \to T$. In [DR2] it was shown that every pure subgroup of a finite-rank Butler group G has the TEP in G. A subgroup S of a torsion-free group G is said to be decent [AH] if to each finite subset X of G there is a finite-rank Butler group B such that $S + B$ is pure and contains X. An important result of [DR2] is that in a countable Butler group G, a pure subgroup H is decent if and only if H has the TEP in G. It is known that a B_2-group G is always a Butler group [BS] and every finite-rank pure subgroup of G is Butler. It is easy to see that if H is a decent subgroup of G and both H and G/H are B_2-groups, then G is also a B_2-group. In the following $\mathbf{Q}$ denotes the additive group of rational numbers and $\mathbf{Z}$ the additive groups of integers. As usual, CH denotes the continuum hypothesis, i.e., $2^{\aleph_0} = \aleph_1$.

3. Pure subgroups of infinite-rank Butler groups

The backbone of the investigation of Butler groups in [DR2] is the theorem that in countable Butler groups the TEP over a pure subgroup implies the decency of the subgroup. In this section we generalize this important result to uncountable Butler groups. Using this, we answer in more than one way a question raised by Arnold [A] and Bican and Salce [BS] for Butler groups of rank $\aleph_1$, namely, whether countable pure subgroups of such groups are again Butler. Using a different approach, this was shown in [D].

We begin our investigation with the following useful lemma.

Lemma 3.1. (i) *Let* $G = \bigcup G_\alpha$ *be a B-filtration of pure subgroups* G_α. *Then* G *has the TEP over each* G_α.

(ii) *Suppose* G *is a* B_2-*group. Then every countable subgroup* H *is contained in a countable pure subgroup* A *which as the TEP in* G.

Proof. (i) Let $f : G_\alpha \to T$ be any homomorphism where T is an arbitrary torsion group. By hypothesis, $G_{\alpha+1} = G_\alpha + B_\alpha$; B_α is pure in G and is a

finite-rank Butler group. By Theorem 2 of [DR2], $f \mid (B_\alpha \cap G_\alpha) : B_\alpha \cap G_\alpha \to T$ extends to a homomorphism $g : B_\alpha \to T$. Then $f_{\alpha+1} : G_{\alpha+1} \to T$ given by $f_{\alpha+1}(x+y) = f(x)+g(y)$ for all $x \in G_\alpha$, $y \in B_\alpha$ is a homomorphism extending f. By transfinite induction, on $\gamma \geq \alpha$, we get an extension $f' : G \to T$.

(ii) By [AH], G has an axiom-3 family $\mathscr{C}$ of decent and pure subgroups. Now $H \subset A \in \mathscr{C}$, where A is countable. The axiom-3 nature of $\mathscr{C}$ and the decency of A enable us to build a B-filtration of G beginning with A. By (i) above, G has the TEP over A.

Lemma 3.1 is useful in proving the following generalization of Theorem 7 of [DR2].

Proposition 3.2. *Let A be a pure subgroup of the torsion-free group G such that*

   (i) *A has TEP in G.*
   (ii) *A is a B_2-group.*
   (iii) *Each finite-rank pure subgroup of G is Butler.*

   Then A is decent in G.

Proof. We may assume without loss of generality that G/A has finite rank. Then $G = H + A$, for some countable pure subgroup H. Since $H \cap A$ is countable and A is a B_2-group, Lemma 3.1(ii) implies that $H \cap A \subseteq B \subseteq A$, B a countable pure subgroup which has the TEP in A and hence in G. Now B has the TEP in the countable Butler group $\langle H + B \rangle_*$ and $\langle H + B \rangle_*/B$ has finite rank. Thus Theorem 7 of [DR2] applies and $\langle H + B \rangle_* = B_1 + B$, where B_1 is a finite-rank Butler group. Then $G = H + A = \langle H + B \rangle_* + A = B_1 + A$ and A is decent in G.

Arnold [A] and Bican, Salce, and Stepan [BS], [BSS] raised the question whether countable pure subgroups of Butler groups are always Butler. Proposition 3.2 enables us to answer this in the affirmative for Butler groups of rank $\aleph_1$. Dugas [D] also answers this for groups of rank $\aleph_1$ using another approach.

Theorem 3.3. (i) *Suppose $G = C/K$ is a Butler group where C is completely decomposable and K is a balanced subgroup of C. If K is a B_2-group, then every countable pure subgroup of G is Butler.*

(ii) *If G is a Butler group of rank $\aleph_1$, then every countable pure subgroup of G is also Butler.*

Proof. (i) Let H/K be a countable pure subgroup of C/K. Since G is Butler, K has the TEP in C and hence in H. Also, any finite-rank pure subgroup of H is Butler. Thus Proposition 3.2 applies and K is decent in H. Since K is balanced in H, we conclude that H/K is the union of an increasing sequence of finite-rank Butler groups and hence is itself Butler; cf. [A].

(ii) Albrecht and Hill [AH] have shown that a balanced subgroup of a completely decomposable group of rank $\aleph_1$ is a B_2-group. Hence (ii) follows from (i).

As another application of Proposition 3.2, we obtain the following (not very satisfactory) characterization of B_2-groups.

Theorem 3.4. *An infinite-rank torsion-free group G is a B_2-group if and only if every finite-rank pure subgroup of G is Butler and $G = \bigcup G_\alpha$ is a smooth filtration of pure subgroups G_α where, for each α, G_α has the TEP in G and $G_{\alpha+1}/G_\alpha$ is countable.*

Proof. We need only prove the sufficiency. We first claim that each G_α is a B_2-group. Suppose, by way of contradiction, there is a β such that G_β is not a B_2-group. We may assume β is the least such ordinal. Then for each $\alpha < \beta$, G_α is a B_2-group and has the TEP in $G_{\alpha+1}$. Then, by Proposition 3.2, G_α is decent in $G_{\alpha+1}$. Since, further, $G_{\alpha+1}/G_\alpha$ is countable, the filtration $\bigcup_{\alpha<\beta} G_\alpha$ can be refined to a B-filtration. This would imply that G_β is a B_2-group, a contradiction. Thus each G_α be a B_2-group and, by Proposition 3.2, must be decent in $G_{\alpha+1}$, Thus the smooth filtration $G = \bigcup G_\alpha$ refines into a B-filtration of G, showing that G is a B_2-group.

Since, for groups of cardinality $\aleph_1$, a smooth filtration of countable groups is an axiom-3 family, we get an axiom-3 characterization of B_2-groups in terms of pure subgroups with the TEP.

Corollary 3.5. *A torsion-free group G is a B_2-group if and only if G satisfies the TEP over an axiom-3 family of pure subgroups and every finite-rank pure subgroup of G is Butler.*

Before we proceed further we review the following proposition, which is a slight recast of Lemma 10 of [DR1] and which is also implicit in the works of Hill and Megibben [HM].

Proposition 3.6. *Suppose C is a completely decomposable group of regular cardinal κ, H is a balanced subgroup C, and $G = C/H$. Then there are a closed and unbounded subset E of κ and smooth filtrations $H = \bigcup_{\alpha \in E} H_\alpha$, $C = \bigcup_{\alpha \in E} C_\alpha$, and $G = \bigcup_{\alpha \in E} G_\alpha$ such that, for each $\alpha \in E$, H_α is balanced in H, C_α is a direct summand of C, G_α is pure in G, $G_\alpha = (C_\alpha + H)/H$, $H_\alpha = H \cap C_\alpha$, and $|H_\alpha| = |C_\alpha| = |G_\alpha| < \kappa$.*

We obtain from Proposition 3.6 an alternative proof of a stronger form of Theorem 3.3. In §7 we will prove a stronger version of 3.7.

Theorem 3.7. *Let $G = C/H$ be a Butler group of regular cardinality κ, where C is completely decomposable and H is balanced.*

(i) If H is a B_2-group, then every countable pure subgroup of G is Butler and G has a smooth filtration of pure subgroups $G = \bigcup G_\alpha$ where each G_α is a Butler group of cardinality $< \kappa$.

(ii) If G is a Butler group of rank $\aleph_1$, then every countable pure subgroup of G is Butler.

Proof. By Proposition 3.6, there are a closed and unbounded subset E of κ and smooth filtrations $H = \bigcup_{\alpha \in E} H_\alpha$, $C = \bigcup_{\alpha \in E} C_\alpha$, and $G = \bigcup_{\alpha \in E} G_\alpha$ with the stated properties. Let $H = \bigcup_{\beta < k} L_\beta$ be a B-filtration of the B_2-group H, so that for all $\beta < k$, $L_{\beta+1} = L_\beta + B_\beta$, where B_β is a Butler group of finite

rank. By Lemma 3.1, H has the TEP over each L_β. Now, by Eklof [E], the two filtrations $\bigcup L_\beta$ and $\bigcup H_\alpha$ agree on a closed and unbounded subset E' of E. Hence we may assume that the $H = \bigcup_{\alpha \in E} H_\alpha$, $C = \bigcup_{\alpha \in E} C_\alpha$, and $G = \bigcup_{\alpha \in E} G_\alpha$ satisfy the conclusion of Proposition 3.6 with the additional property that H has the TEP over each H_α. Since G is Butler, C has the TEP over H. Then H and hence C_α will have the TEP over H_α. Since H_α is a B_2-group, Proposition 3.2 implies that H_α is decent in C_α. Since H_α is balanced in C_α, we conclude that G_α is a Butler group in which every finite-rank pure subgroup is Butler.

A natural question is whether the converse of Theorem 3.7 (i) holds. Observe that if $G = \bigcup_{\alpha \in E} G_\alpha$ is a smooth filtration of pure Butler subgroup G_α and each countable pure subgroup of G is Butler, then Proposition 3.6 provides a closed and unbounded subset E' of E and a filtration $H = \bigcup_{\alpha \in E'} H_\alpha$ where each H_α is balanced in H, $H_\alpha = H \cap C_\alpha$, and $C_\alpha/H_\alpha \cong G_\alpha$, so that C_α has the TEP over H_α. By Proposition 3.2, H_α is decent in C_α and hence in H. Thus $H = \bigcup_{\alpha \in E'} H_\alpha$ is a smooth filtration where, for each α, H_α is balanced, decent, and has the TEP in H and every countable subgroup of H is Butler. Moreover, H itself is decent in C. Is H a B_2-group? When does H fit into a B-filtration of C? These are the questions that we wish to explore next.

Lemma 3.8. *Suppose A and B are pure subgroups of G with $A \subseteq B$. If A and B/A have the TEP respectively in G and G/A, then B has the TEP in G.*

Proof. Let T be any torsion group and $f \in \text{Hom}(B, T)$. Now $f \mid A : A \to T$ extends to a homomorphism $h : G \to T$. Since $h - f$ vanishes on A, $h - f \mid B$ induces $h_1 : B/A \to T$, which, by hypothesis, extends to a $g_1 \in \text{Hom}(G/A, T)$. Let g_2 be the natural map $G \to G/A$ followed by g_1. Then $g = h - g_2$ satisfies, for all $b \in B$, $g(b) = h(b) - g_2(b) = f(b)$.

Proposition 3.9. *Let G be a torsion-free group such that any finite-rank pure subgroup is Butler. Suppose A is a balanced subgroup or A is a B_2-group. Then A fits into a B-filtration of G; i.e., $G = \bigcup_{\alpha < \lambda} G_\alpha$ with $G_0 = A$, $G_{\alpha+1} = G_\alpha + B_\alpha$, B_α finite-rank Butler if and only if A has the TEP in G and G/A is a B_2-group.*

Proof. Clearly, if A fits into a B-filtration with $A = G_0$, $G_{\alpha+1} = G_\alpha + B_\alpha$, B_α finite-rank Butler, and $G = \bigcup_{\alpha < \lambda} G_\alpha$, then $\bigcup_{\alpha < \lambda}(G_\alpha/A)$ is a B-filtration of G/A which is therefore a B_2-group. By Lemma 3.1, A will have the TEP in G.

Conversely, Let $G/A = \bigcup_{\alpha < \lambda}(G_\alpha/A)$ be a B_2-filtration so that, for each α, $G_{\alpha+1}/A = G_\alpha/A + B_\alpha/A$ where B_α/A is finite-rank Butler. If A is balanced then $B_\alpha = A + B'_\alpha$, where B'_α is finite-rank Butler, and so $G_{\alpha+1} = G_\alpha + B'_\alpha$, proving that $\bigcup G_\alpha$ with $G_0 = A$ is a B_2-filtration of G from G_0 onwards. If, on the other hand, A is a B_2-group, then Proposition 3.2 applies so that A is decent in G. Thus, as before, $B_\alpha = A + B'_\alpha$, B'_α finite-rank Butler. Then $\bigcup G_\alpha$ with $G_0 = A$ is a B_2-filtration of G and this second-case G itself is a B_2-group.

Corollary 3.10. *Let* $B = \bigcup_{\alpha<\lambda} B_\alpha$ *be a smooth filtration with* $B_0 = 0$ *and, for each* α, B_α *is pure and has the TEP in* B. *If* $B_{\alpha+1}/B_\alpha$ *is a* B_2*-group for all* α, *then* B *itself is a* B_2*-group.*

The following proposition provides a partial answer to the question raised earlier.

Proposition 3.11. *Suppose* $0 \to H \to C \to G \to 0$ *is a balanced exact sequence of torsion-free groups of cardinality* $\aleph_n$, $n < \omega$. *If* C *and* G *are* B_2*-groups, then so is* H.

Proof. Identify H with a subgroup of C and take $G = C/H$. We will do an induction on n, the result being true for $n = 0$. Suppose $G = \bigcup_{\alpha<\omega_n} G_\alpha$, $C = \bigcup_{\alpha<\omega_n} C_\alpha$ are B-filtrations of G and C, respectively, so that for each α, G_α and C_α are decent and have the TEP respectively in G and C; for $\alpha < \beta$, G_β/G_α and C_β/C_α are B_2-groups, G_α and C_α are B_2-groups themselves, and $|C_\alpha| = |G_\alpha| < \aleph_n$. The proof of Proposition 3.6 shows that there are closed and unbounded subset E of ω_n and smooth filtrations $H = \bigcup_{\alpha\in E} H_\alpha$, $C = \bigcup_{\alpha\in E} C_\alpha$ $G = \bigcup_{\alpha\in E} G_\alpha$ such that $H_\alpha = H \cap C_\alpha$, H_α balanced in C_α, $G_\alpha = (H+C_\alpha)/H \cong C_\alpha/H_\alpha$. Since G_α is a B_2-group (and hence Butler) and H_α is balanced in C_α, H_α has the TEP in C_α and hence in C. By induction, H_α is a B_2-group for each α. Now $H_{\alpha+1}/H_\alpha$ is a balanced subgroup of $C_{\alpha+1}/H_\alpha$. Moreover, $C_{\alpha+1}/H_\alpha$ is a decent extension of the B_2-group C_α/H_α by the B_2-group $C_{\alpha+1}/C_\alpha$ and hence is itself a B_2-group. Since $(C_{\alpha+1}/H_\alpha)/(H_{\alpha+1}/H_\alpha) \cong C_{\alpha+1}/H_{\alpha+1} \cong G_{\alpha+1}$ is a B_2-group, cardinal induction implies that $H_{\alpha+1}/H_\alpha$ is a B_2-group.

4. PROPERTIES OF $\text{Bext}(-, T)$ AND h-MAPS

It is well known [Hu] that if $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is balanced exact, then for any torsion T, $0 \to \text{Hom}(C, T) \to \text{Hom}(B, T) \to \text{Hom}(A, T) \xrightarrow{\delta} \text{Bext}(C, T) \xrightarrow{\beta^*} \text{Bext}(B, T) \xrightarrow{\alpha^*} \text{Bext}(A, T)$ is exact. We want to investigate what can be said without the balancedness of $0 \to A \to B \to C \to 0$.

The concept of an h-map is introduced and its relationship to separable subgroups and, in particular, balanced subgroups is investigated. The notion of an h-map appears to have the potential to play a dominant role in the future investigations of infinite-rank Butler groups.

Lemma 4.1. *Consider the following row exact commutative diagram:*

$$
\begin{array}{ccccc}
0 \to T \to M' & \xrightarrow{\eta} & G & \to 0 \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \beta} & \\
0 \to T \to M & \xrightarrow{\alpha} & G/A & \to 0 \\
& & \downarrow & \\
& & 0 &
\end{array}
$$

where the right square is a pullback, T any group and G/A torsion-free. If the bottom row is balanced exact, then the top row is balanced exact.

Proof. Let R be a torsion-free group of rank 1 and $\varphi : R \to G$ a homomorphism. Since the bottom row is balanced, there is a map $\psi : \beta(\varphi(R)) \to M$ with $\beta\varphi = \alpha\psi$. Thus there exists a map $\gamma : R \to M'$ with $\eta\gamma = \varphi$ since M' is a pullback. This shows that the top row is balanced.

The following is an immediate consequence:

Corollary 4.2. *The map $\beta : G \to G/A$, G/A torsion-free, induces a map*

$$\beta^* : \mathrm{Bext}(G/A, T) \to \mathrm{Bext}(G, T)$$

for any torsion group T.

We will need the following later:

Corollary 4.3. *If G is a Butler group and has the TEP over a pure subgroup A, then G/A is a Butler group.*

Proof. Since A has the TEP over G, the following is exact:

$$0 \to \mathrm{Ext}(G/A, T) \xrightarrow{\beta^*} \mathrm{Ext}(G, T).$$

Since G/A is torsion-free we may apply Corollary 4.2 and obtain

$$0 \to \mathrm{Bext}(G/A, T) \xrightarrow{\beta^*} \mathrm{Bext}(G, T)$$

is exact. Thus $\mathrm{Bext}(G/A, T) = 0$ and G/A is a Butler group. $\square$

If $A \subseteq G$ is as above, we want to investigate when

$$\mathrm{Bext}(G, T) \xrightarrow{\alpha^*} \mathrm{Bext}(A, T) \to 0$$

is exact for some $0 \to A \xrightarrow{\alpha} G$. This turns out to be quite involved.

First we need some notation. Let T be a reduced torsion group and A reduced torsion-free. We will fix a mixed group M that fits into $0 \to T \to M \xrightarrow{\pi} A \to 0$. For any group G, let $\widehat{G}$ be the cotorsion hull of G; see [F1]. Then G is a pure subgroup of $\widehat{G}$ and $\widehat{G}/G$ is divisible. We obtain a commutative diagram with exact rows:

$$0 \to \widehat{T} \to \widehat{M} \underset{\rho}{\overset{\hat{\pi}}{\rightleftarrows}} \widehat{A} \to 0,$$

$$\cup \quad \cup \quad \cup$$

$$0 \to T \to M \xrightarrow{\pi} A \to 0.$$

$\hat{\pi}$ is the map induced by π on the cotorsion hull and ρ is a splitting map, i.e., $\hat{\pi}\rho = \mathrm{id}_{\widehat{A}}$. We will use this to obtain a description of M as a subgroup of $\widehat{M}$: There is a map $\varphi : A \to \widehat{T}$ such that $M = T + \langle \rho(b) + \varphi(b) \mid b \in A \rangle$. Here $\varphi : A \to \widehat{T}$ is a map such that $\varphi^* : A \to \widehat{T}/T$ defined by $\varphi^*(x) = \varphi(x) + T$ is a homomorphism. If we pick another splitting map $\rho' : \widehat{A} \to \widehat{M}$, then again $M =$

$T + \langle \rho'(b) + \varphi'(b) \mid b \in A \rangle$ and an easy computation shows $\varphi(b) \equiv \varphi'(b) \bmod T$ for all $b \in A$. We will call such maps φ and φ' *equivalent* since they yield the same mixed group M in the way described above. We find the maps $\varphi : A \to \widehat{T}$ to be of some help since they allow us to "code" most of the structure of M into a single map. In what follows we will omit the splitting map ρ; i.e., we will set $\rho = \mathrm{id}_{\widehat{A}}$. Since we are interested in the elements of $\mathrm{Bext}(A, T)$ we will investigate what maps $\rho : A \to \widehat{T}$ belong to mixed groups M with T balanced in M.

Definition. Let A be a torsion-free group. A map $\varphi : A \to \widehat{T}$ is called an *h-map* if

> (j)   The composite $a \mapsto \varphi(a) + T$ is a homomorphism from A into $\widehat{T}/T$.
> (ij) If $|a|_p = \infty$, then $\varphi(a)_p \in T_p + p^\infty \widehat{T}_p$.
> (iij) For almost all primes p, $|a|_p \le |\varphi(a)|_p$ for any $a \in A$.

We call two h-maps φ, $\psi : A \to \widehat{T}$ *equivalent* if for all $a \in A$, $\varphi(a) \equiv \psi(a) \bmod T$. Note that each h-map is equivalent to an h-map ψ and $|\psi(a)|_p \ge |a|_p$ for all $a \in A$ and all primes p.

The next proposition relates h-maps from A into $\widehat{T}$ to elements in $\mathrm{Bext}(A, T)$. There we will use the following observation: If X, Y are pure subgroups of some group G containing its torsion part T with $X \cap Y = T$ and Y/T divisible, then $X + Y$ is pure in G. We omit the straightforward proof.

Proposition 4.4. *Let T be a reduced torsion group and B reduced torsion-free.*
  (j) *If $E : 0 \to T \to M \to B \to 0$ is exact, then E is balanced exact if and only if there is an h-map $\varphi : B \to \widehat{T}$ such that as a subgroup of $\widehat{M} = \mathrm{Ext}(\mathbf{Q}/\mathbf{Z}, M) \cong \widehat{T} \oplus \widehat{B}$, $M = T + \langle \{b + \varphi(b) \mid b \in B\} \rangle$.*
  (jj) *If the short exact sequence E is balanced exact, then E splits iff the h-map φ is equivalent to a homomorphism $\varphi : B \to \widehat{T}$, i.e., $\varphi(b) + T = \varphi(b) + T$ for all $b \in B$.*

Proof. (j) Let E be balanced exact. The mixed group M is a pure subgroup of its cotorsion hull $\widehat{M} = \widehat{T} \oplus \widehat{B}$. Since $\widehat{T}/T$ is divisible and M/T reduced, $M \cap \widehat{T} = T$. If $\pi : \widehat{M} \to \widehat{B}$ is the natural projection along $\widehat{T}$, then $\pi(M) = B$. The remark above gives that $M + \widehat{T}$ is pure in $\widehat{M}$ and hence $B = (M + \widehat{T})/\widehat{T}$ is pure in $\widehat{M}/\widehat{T} = \widehat{B}$. Since T is balanced in M, to each $b \in B$ there exists $m_b \in M$ such that $|m_b| = |b|$ and $m_b + T = b$. We may write $m_b = \pi'(m_b) + b$, where $\pi' : \widehat{M} \to \widehat{T}$ is the projection. Clearly, $|\pi'(m_b)| \ge |m_b| = |b|$. Then the map $\varphi : B \to \widehat{T}$ defined by $\varphi(b) = \pi'(m_b)$ satisfies $|\varphi(b)| \ge |b|$ and the induced map $\overline{\varphi} : b \mapsto \varphi(b) + T$ is a homomorphism. Thus φ is an h-map and $M = T + \langle m_b : b \in B \rangle = T + \langle \{\varphi(b) + b : b \in B\} \rangle$.

Conversely, suppose $0 \to T \to M \to B \to 0$ is exact and there is an h-map $\varphi : B \to \widehat{T}$ with $M = T + \langle \varphi(b) + b \mid b \in B \rangle \subseteq \widehat{M}$. Then M is a mixed group with $M/T \cong B$. If $0 \ne b \in B$, then $|\varphi(b)|_p \ge |b|_p$ for almost all p. If $|\varphi(b)|_p < |b|_p$, then $|b|_p < \infty$ because φ is an h-map. Since

$\varphi(b)_p \in \hat{T}_p$ is divisible $\bmod T_p$ and there are only finitely many such primes with $|\varphi(b)|_p < |b|_p$, we find $t \in T$ with $|t + \varphi(b)|_p \geq |b|_p$ for *all* primes p. Thus $\varphi(b) + t$ is proper in $\varphi(b) + T$ and T is balanced in M. The proof of (jj) is straightforward.

The following theorem is crucial for this paper. It will enable us to discover many subgroups of Butler groups that are again Butler groups.

Theorem 4.5. *Let A be a pure and separable subgroup of the torsion-free group B with $|B/A| \leq \aleph_1$. Let $\varphi : A \to \hat{T}$ be an h-map and assume that each pure subgroup of finite rank of B is Butler. Then there exists an h-map $\psi : B \to \hat{T}$ with $\psi \mid A = \varphi$.*

Proof. We may assume that B/A is countable. For each coset $b + A \in B/A$ there exists a countable subset $C = C(b) \subseteq A$ such that for each $a \in A$ there exists an element $a_0 \in C$ with $|b + a| \leq |b + a_0|$. Let B_0 be the pure and countable (!) subgroup generated by a representing system of the cosets in B/A together with the countably many elements in $C = C(b)$, b running over the representing system of B/A. Let $A_0 = A \cap B_0$. Then A_0 is pure in A and our hypotheses imply that A_0 and B_0 are (countable) Butler groups. Moreover, $B = A + B_0$. Since A_0 is a Butler group we may apply 4.4(jj) and assume that $\varphi \mid A_0 \in \mathrm{Hom}(A_0, \hat{T})$. Since $\hat{T}$ is pure-injective there is a $\psi \in \mathrm{Hom}(B_0, \hat{T})$ with $\psi \mid A_0 = \varphi \mid A_0$. For each pair $(b_0, a) \in B_0 \times A$ we fix $a_0 = a_0(b_0, a) \in A_0$ with $|b_0 + a| \leq |b_0 + a_0|$. Note that such an $a_0 \in A_0$ exists by the choice of A_0.

Now define $\rho(b_0 + a) = \psi(b_0 + a_0) + \varphi(a - a_0)$. Then $\rho : B \to \hat{T}$ is a map. Suppose $b = b_0 + a \in B_0$. Then $a \in A_0$ and we have $\rho(b_0 + a) = \psi(b_0 + a_0) + \varphi(a - a_0) = \psi(b_0 + a)$ since ψ is a homomorphism with $\psi \mid A_0 = \varphi \mid A_0$. Thus $\rho \mid B_0 = \psi$. If $b_0 + a \in A$, then $b_0 \in B_0 \cap A = A_0$ and

$$\rho(b_0 + a) = \psi(b_0 + a_0) + \varphi(a - a_0)$$
$$= \varphi(b_0 + a_0) + \varphi(a - a_0) \equiv \varphi(b_0 + a) \quad \bmod T.$$

Thus $\rho \mid A$ is equivalent to φ and for $M = T + \{(\varphi(a), a) \mid a \in A\}$, $M' = T + \{(\rho(b), b) \mid b \in B\}$ we have that the diagram

$$0 \to T \to M \ \to A \to 0$$
$$\| \quad\quad \cap \quad\quad \cap$$
$$0 \to T \to M' \to B \to 0$$

is commutative. In order to finish the proof we have only to show that ρ is an h-map. (After that we may change ρ into an equivalent h-map ρ' with $\rho' \mid A = \varphi$.) Let $b \in B$, $b = b_0 + a$, $b_0 \in B_0$, $a \in A$. Then $|b| = |b_0 + a| \leq |b_0 + a_0|$ where $a_0 \in A_0$. We obtain

$$|\rho(b_0 + a)|_p \geq \min\{|\psi(b_0 + a_0)|_p, |\varphi(a - a_0)|_p\} \geq \min\{|b_0 + a_0|_p, |a - a_0)|_p$$

for almost all primes p. If q is a prime with $|b_0 + a|_q = \infty$, then $|b_0 + a_0|_q = \infty = |a - a_0|_q$ and $\varphi(a - a_0)_q$, $\psi(b_0 + a_0)_q \in T_q + q^\infty \hat{T}_q$. Thus $\rho(b_0 + a)_q \in T_q + q^\infty \hat{T}_q$. This completes the proof.

Corollary 4.6. *Let B be a Butler group such that for any countable subset X of B there is a filtration $B = \bigcup_{\alpha < \lambda} B_\alpha$ such that $|B_{\alpha+1}/B_\alpha| \le \aleph_1$, $X \subseteq B_0$, $|B_0| \le \aleph_1$, and each B_α is a separable subgroup of $B_{\alpha+1}$. Then each B_α is a Butler group.*

Proof. Let A be a pure subgroup of finite rank of B. Then we have a filtration $B = \bigcup_{\alpha < \lambda} B_\alpha$ such that each B_α is separable in $B_{\alpha+1}$ and $B_0 = A$. By [D, Theorem 2] and [B] (see also [A]) we conclude that A is a Butler group. Thus any pure countable subgroup of B is Butler. Let T be torsion and let $0 \to T \to M_\alpha \to B_\alpha \to 0$ be balanced exact. By Theorem 4.5, there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & M_\alpha & \longrightarrow & B_\alpha & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & M_{\alpha+1} & \longrightarrow & B_{\alpha+1} & \longrightarrow & 0
\end{array}
$$

where $B_\alpha \to B_{\alpha+1}$ is the embedding and the bottom row is balanced exact. A transfinite induction provides a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & M_\alpha & \longrightarrow & B_\alpha & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & B & \longrightarrow & 0
\end{array}
$$

where the bottom row is balanced exact. Since B is Butler, $M = T \oplus B'$ for some subgroup B' isomorphic to B. Since $T \subseteq M_\alpha$ we obtain

$$M_\alpha = M \cap M_\alpha = (T + B') \cap M_\alpha = T \oplus (B' \cap M_\alpha).$$

Thus M_α splits and B_α is a Butler group. $\square$

Example. A group B with pure subgroup A, $\mathrm{rk}(B/A) = 1$ and there is an h-map $\varphi : A \to T = \prod_p \hat{T}_p$ which does not lift to B.

Construction. Let $\mathscr{F}$ be a maximal family of almost disjoint maps $f : \omega \to \omega$, i.e., $f \ne g \in \mathscr{F} = \{i \mid f(i) = g(i)\}$ finite.

Generators. x, x_n $(n < \omega)$, x_f $(f \in \mathscr{F})$.

Define $B = \langle x, x_n, x_f \mid n < \omega, f \in \mathscr{F} \rangle$.

Let π be an infinite set of primes,

$$\pi = \{p_{ij} \mid i, j < \omega\}.$$

Relations. $(x + x_n)/p_{n,i}$ $(i < \omega)$ and $(x + x_f)/p_{n,f(n)}$ $(n < \omega)$, i.e.,

$$B = \left\langle x, x_n, x_f, \frac{x + x_n}{p_{n,i}}, \frac{x + x_f}{p_{n,f(n)}} \mid n < \omega, i < \omega, f \in \mathscr{F} \right\rangle.$$

Let $A = \langle x_n, x_f \mid n < \omega, f \in \mathscr{F} \rangle_*$. Then

$$A = \left\langle x_n, x_f, \frac{x_n - x_f}{p_{n,f(n)}} \frac{x_f - x_g}{p_{n,f(n)}} \mid n < \omega, g, f \in \mathscr{F}, f(n) = g(n) \right\rangle.$$

Pick elements $s_{ij}, t_{ij} \in T_{p_{ij}}$ with $|s_{ij} - t_{ij}|_{p_{ij}} = 0$.
Define

$$\varphi(x_n)_{p_{ij}} = \begin{cases} s_{ij} & \text{if } i = n, \\ 0 & \text{if } i \neq n \end{cases}$$

and

$$\varphi(x_f)_{p_{ij}} = \begin{cases} t_{n,f(n)} & \text{if } (i,j) = (n,f(n)), \\ 0 & \text{elsewhere}. \end{cases}$$

φ induces a homomorphism $\varphi' : A' = \langle x_n, x_f \mid n < \omega, f \in \mathscr{F} \rangle \to \widehat{T}/T$.
Since $\widehat{T}/T$ is divisible, there is $\varphi'' : A \to \widehat{T}/T$ extending φ'. If $a \in A$,
$a \notin \{x_n, x_f \mid n < \omega, f \in \mathscr{F}\}$, define $\varphi(a)$ to be any element of the coset
$\varphi''(a)$. Since A is homogeneous of type 0, φ is an h-map, $\varphi : A \to \widehat{T}$.

   Now suppose φ lifts to an h-map $\psi : B \to \widehat{T}$. Then $|\psi(x) + \varphi(x_n)|_{p_{n,1}} > 0$ for all $i \geq m_0(n)$, i.e., $\psi(x)_{p_{n,1}} \equiv s_{ni} \mod p_{ni}\widehat{T}_{p_{n,1}}$ for $i \geq m_0(n)$; on
the other hand, define $g : \omega \to \omega$ via $g(i) = m_0(i) \Rightarrow \exists f \in \mathscr{F}$ such that
$W = \{i < \omega \mid f(i) = g(i)\}$ is infinite. We have $|\psi(x) + \varphi(x_f)|_{p_{n,f(n)}} \geq 1$ for
$n \geq k_0(f) \Rightarrow \psi(x)_{p_{n,f(n)}} \equiv t_{n,f(n)} \mod p_{n,f(n)}\widehat{T}_{p_1}$. Since ω is infinite, there is
$i \in W$, $i \geq k_0(f)$, and for $n \geq k_0(f)$ we have

$$\psi(x)_{p_{n,i}} \equiv s_{ni} \mod p_{ni}\widehat{T}_{p_{n,i}} \quad \text{and}$$

$$\psi(x)_{p_{n,i}} \equiv t_{ni} \mod p_{ni}\widehat{T}_{p_{n,i}}.$$

This implies $s_{ni} \equiv t_{ni} \mod p_{ni}\widehat{T}_{p_{ni}}$, which is a contradiction to the choice of
s_{ni} and t_{ni}.

5. Separable subgroups of torsion-free groups

   Our goal in this section is to prove the following

Theorem 5.1. *Assume CH holds. If G is a torsion-free group of cardinality $\aleph_n$,*
$n \leq \omega$, then there is a smooth chain of separable pure subgroups G_α, $\alpha < \aleph_n$,
such that $G = \bigcup_{\alpha < \aleph_n} G_\alpha$ and $G_{\alpha+1}/G_\alpha$ has rank 1.

   We will prove this theorem after some preparation. First we need a

Definition. Let H be a pure subgroup of the torsion-free group G. Let $g \in G$.
For each countable subset C of H, we define a map $g_c^\#$ from C into the set
of all height sequence by $g_c^\#(x) = |g + x|$ for all $x \in C$. We H *hyperbalanced*
if for each $g \in G$ and for each countable subset C of H there is some $h \in H$
with $g_c^\# = h_c^\#$.

Lemma 5.2. *Let $0 \neq H$ be a subgroup of G. Then there is a hyperbalanced*
subgroup L of G with $H \subseteq L$ and $|L| \leq |H|^{\aleph_0}$.

Proof. By induction on $\alpha < \omega_1$ we define a smooth chain of subgroups H_α with
$H_0 = H$ and $H_\lambda = \bigcup_{\alpha < \lambda} H_\alpha$ if $\lambda < \omega_1$ is a limit ordinal. Moreover, we will

have $|H_\alpha| \le |H|^{\aleph_0}$. Suppose H_α is already defined. Then $S = \{g_C^\# \mid g \in G, \, C$ a countable subset of $H_\alpha\}$ has cardinality $|H_\alpha|^{\aleph_0} \cdot (2^{\aleph_0})^{\aleph_0} \le (|H|^{\aleph_0})2^{\aleph_0} = |H|^{\aleph_0}$. Thus there is a subset W of G with $|W| \le |H|^{\aleph_0}$ and $S = \{g_C^\# \mid g \in W, \, C$ a countable subset of $H\}$. Now define $H_{\alpha+1}$ to be the pure subgroup generated by H_α and S. Note that $|H_{\alpha+1}| \le |H|^{\aleph_0}$. We will show that $L = \bigcup_{\alpha < \omega_1} H_\alpha$ is hyperbalanced in G: Let C be a countable subset of L. Since $cf(\omega_1) = \omega_1 > \omega$ there is $\alpha < \omega_1$ such that $C \subseteq H_\alpha$. If $g \in G$ then the construction of $H_{\alpha+1}$ shows the existence of an $h \in H_{\alpha+1} \subseteq L$ with $g_C^\# = h_C^\#$. Thus L is hyperbalanced and $|L| \le \omega_1 \cdot |H|^{\aleph_0} = |H|^{\aleph_0}$. $\square$

The argument above also shows

Observation 5.3. If $\{H_\alpha, \alpha < \lambda\}$ is a smooth chain of subgroups of G, $cf(\lambda) > \omega$, and each $H_{\alpha+1}$ is hyperbalanced in G, then $\bigcup_\alpha H_\alpha$ is hyperbalanced in G.

The following is crucial:

Lemma 5.4. *Any hyperbalanced subgroup H of G is balanced in G.*

Proof. Assumed not. Then there is a coset $0 \ne g + H$ without proper element. By induction on $\alpha < \omega_1$ we will construct elements $g_\alpha \in g+H$ with $|g_\alpha| < |g_\beta|$ for $\alpha < \beta$. Let g_0 be any element in $g + H$ and suppose that g_γ has been defined for all $\gamma < \alpha$.

Note that $C = \{g_\gamma - g \mid \gamma < \alpha\}$ is countable. Since H is hyperbalanced, there is some $h_1 \in H$ with $g_C^\# = (h_1)_C^\#$, i.e., $|g + (g_\gamma - g)| = |h_1 + (g_\gamma - g)|$ for all $\gamma < \alpha$. This implies $|g - h_1| \ge |g_\gamma|$ for all $\gamma < \alpha$. Since there is no proper element in $g + H$, $g - h_1$ is not proper and there is some $h_2 \in H$ with $|g + h_2| \not\le |g - h_1|$. Set $C' = C \cup \{-h_1, h_2\}$ and use again that H is hyperbalanced and we find $h_3 \in H$ with $|g + h_3| \ge |g + h_2|$, $|g - h_1|$, and $|g_\gamma|$ for all $\gamma < \alpha$, Now suppose $|g + h_3| = |g_\gamma|$ for some $\gamma < \alpha$. Since $|g + h_3| \ge |g - h_1| \ge |g_\gamma|$ we obtain

$$|g + h_2| \le |g + h_3| = |g_\gamma| \le |g - h_1| \not\le |g + h_2|,$$

a contradiction. This show $|g+h_3| > |g_\gamma|$ for all $\gamma < \alpha$. We may set $g_\alpha = g+h_3$ and the induction works. This also completes the proof of the lemma, since there is no *uncountable* ascending chain of height sequences. $\square$

Let G be a torsion-free group. By induction on ordinals $\alpha < \omega_1 \ (= \aleph_1)$ we define B^α-subgroups of G: The B^0-subgroups of G are precisely the balanced subgroups. If $\lambda < \omega_1$ is a limit ordinal, then the B^λ-subgroups are all the subgroups of G that are B^α-subgroups for some $\alpha < \lambda$. The $B^{\alpha+1}$-subgroups are the unions of countable chains of B^α-subgroups. We call a subgroup A of G a B^∞-subgroup if A is a B^α-subgroup for some $\alpha < \omega_1$.

Lemma 5.5. (i) *The class of B^∞-subgroups is closed with respect to unions of countable chains.*

(ii) *If A is a B^∞-subgroup of B and B is balanced in C, then A is a B^∞-subgroup of C.*

Proof. (i) follows since $cf(\omega_1) > \omega$. To prove (ii) we use induction. If A is a B^0-subgroup of B, i.e., A is balanced in B, then A is balanced in C since "balanced" is transitive. We may assume that A is a $B^{\alpha+1}$-subgroup of B and (ii) is true for B^α-subgroups. Now $A = \bigcup_{n<\omega} A_n$, A_n, B^α-subgroups of B. Thus A_n are B^∞-subgroups of C and A is a B^∞-subgroup of C.

Theorem 5.6. *Assume CH holds. Let G' be a torsion-free group such that G, H_m, $G + H_m$ ($m < \omega$) are balanced subgroups of G'. Let $|G| = \aleph_n$, $2 \leq n < \omega$. Then there is an $\aleph_n$-filtration $G = \bigcup_{\alpha<\aleph_n} G_\alpha$ of G, $|G_\alpha| = \aleph_{n-1}$, and $G_\alpha(G_\alpha + H_m)$ is a B^∞-subgroup of $G(G + H_m)$ for all α and $|G_{\alpha+1}/G_\alpha| \leq \aleph_1$. If X is a countable subset of G, then we may have that $X \subseteq G_0$.*

Proof. By induction on n. We may assume $H_0 = 0$. Suppose $|G| = \aleph_2$. Lemma 5.2, Observation 5.3, and a standard back-and-forth argument provide a filtration $G = \bigcup_{\alpha<\aleph_2} G_\alpha$ where $|G_\alpha| = \aleph_1$, $X \subseteq G_0$, and $(G_\alpha + H_m)/H_m$ is hyperbalanced in $(G + H_m)/H_m$ for all $m < \omega$ whenever $cf(\alpha) \neq \omega$. Here we had to use CH to obtain $\aleph_n^{\aleph_0} = \aleph_n$ in order to avoid possible jumps of our cardinals in 5.2. Since hyperbalanced implies balanced (Lemma 5.4), we have for $cf(\alpha) \neq \omega$: $G_\alpha + H_m$ is balanced in $G + H_m$. Thus for all $\alpha < \aleph_2$, G_α (resp. $G_\alpha + H_m$) is a B'-subgroup of G (resp. $G + H_m$). Since $|G_\alpha| = \aleph_1$, $|G_{\alpha+1}/G_\alpha| \leq \aleph_1$ and the theorem holds for $n = 2$. Now suppose the theorem holds for n. Let $|G| = \aleph_{n+1}$. As before, we get an $\aleph_{n+1}$-filtration $G = \bigcup_{\alpha<\aleph_{n+1}} G_\alpha$ where $(G_\alpha + H_m)/H_m$ are hyperbalanced in $(G + H_m)/H_m$ whenever $cf(\alpha) \neq \omega$. Thus $G_\alpha + H_m$ is balanced in $G + H_m$ whenever $cf(\alpha) \neq \omega$. We now show that we can refine this filtration to meet the requirements of our theorem. We distinguish two cases.

(I) $cf(\alpha) \neq \omega$. Here we apply the induction hypothesis where $G_{\alpha+1}$ plays the role of G and G_α, $G_\alpha + H_m$ play the role of H_m. Thus we find A_r, $r < \aleph_n$, and A_r is a B^∞-subgroup of $G_{\alpha+1} + (G_\alpha + H_m)$ and $A_r + G_\alpha$ is a B^∞-subgroup of $G_{\alpha+1} + G_\alpha$. Now set $G_{\alpha,r} = G_\alpha + A_r$. Then $G_{\alpha,r}$ is a B^∞-subgroup of $G_{\alpha+1}$ and $H_m + G_{\alpha,r}$ is a B^∞-subgroup of $G_{\alpha+1} + H_m$. Since $G_{\alpha+1}$, $G_{\alpha+1} + H_m$ are balanced in G', $H_m + G_{\alpha,r}$ (resp. $G_{\alpha,r}$) is a B^∞-subgroup of G' by Lemma 5.5. Since $|A_{r+1}/A_r| \leq \aleph_1$ we conclude that also $|G_{\alpha,r+1}/G_{\alpha,r}| \leq \aleph_1$.

(II) $cf(\alpha) = \omega$. Here $G_\alpha = \bigcup_{n<\omega} G_{\alpha_n}$ and the α_n are in case I. We apply the induction hypothesis where $G_{\alpha+1}$ plays the role of G and G_{α_n}, $H_m + G_{\alpha_n}$ play the role of the H_m, $(m, n) \in \omega \times \omega$. Again, we find an $\aleph_n$-filtration $G_{\alpha+1} = \bigcup_{r<\aleph_n} A_r$ where $|A_{r+1}/A_r| \leq \aleph_1$, $A_r + H_m + G_{\alpha_n}$ is a B^∞-subgroup of $G_{\alpha+1} + H_m$, and $A_r + G_{\alpha_n}$ is a B^∞-subgroup of $G_{\alpha+1}$. This implies that $\bigcup_{n<\omega}(A_r + G_{\alpha_n}) + H_m$ is a B^∞-subgroup of $G_{\alpha+1} + H_m$ and $\bigcup_{n<\omega} A_r + G_{\alpha_n}$ is a B^∞-subgroup of $G_{\alpha+1}$. Thus we may set $G_{\alpha,r} = \bigcup_{n<\omega}(A_r + G_{\alpha_n})$ and because of Lemma 5.5, $G_{\alpha,r}$ is a B^∞-subgroup of G' and $H_m + G_{\alpha,r}$ is a B^∞-subgroup of

G'. Again, since $|A_{r+1}/A_r| \leq \aleph_1$, we obtain $|G_{\alpha,r+1}/G_{\alpha,r}| \leq \aleph_1$. This finishes the proof of the theorem. □

Corollary 5.7. *Assume CH holds. Let G be a torsion-free group, $|G| \leq \aleph_\omega$, and X a countable subset of G. Then G has a filtration $G = \bigcup_{\alpha < \kappa} G_\alpha$ with $G_{\alpha+1}/G_\alpha$ torsion-free of rank 1, G_α separable in G, and $X \subseteq G_0$, where G_0 is countable.*

Proof. First, let $|G| = \aleph_n < \aleph_\omega$. If $|G| \leq \aleph_1$, there is nothing to show since each countable subgroup is separable. If $|G| = \aleph_n$, we have only to use Theorem 5.6 and observe that B^∞-subgroups are separable and countable extensions of separable subgroups are again separable. If $|G| = \aleph_\omega$, we may use Lemma 5.2, Observation 5.3, and Lemma 5.4 to write $G = \bigcup_{n < \omega} G_n$, G_n balanced in G, $|G_n| = \aleph_n$. We apply Theorem 5.6 to G_{n+1}/G_n and find $G_n \subset A_{n,r} \subset G_{n+1}$ with $A_{n,r}/G_n$ a B^∞-subgroup of G_{n+1}/G_n. We want to show that this implies that $A_{n,r}$ is a B^∞-subgroup of G_{n+1}.

If $A_{n,r}/G_n$ is a B^0-subgroup of G_{n+1}/G_n, then $A_{n,r}/G_n$ is simply balanced in G_{n+1}/G_n, and since G_n is balanced in G_{n+1}, we have that $A_{n,r}$ is balanced. Now assume that $A_{n,r}/G_n = \bigcup_{i<\omega}(A_i/G_n)$ is a $B^{\alpha+1}$-subgroup of G_{n+1}/G_n. Our induction implies that A_i is a B^∞-subgroup of G_{n+1}; i.e., $A_{n,r} = \bigcup_{i<\omega} A_i$ is a B^∞-subgroup of G_{n+1}. Thus $G = \bigcup_{n,r} A_{n,r}$ is a filtration into B^∞-subgroups of G with $|A_{n,r+1}/A_{n,r}| \leq \aleph_1$. This filtration may be refined to prove the theorem.

Our methods break down if $|G| = \aleph_{\omega+1}$, since one cannot show in general that there are many or even one balanced subgroup of cardinality $\aleph_\omega$ of G when $|G| = \aleph_{\omega+1}$, as the following example shows. Again, we assume $\aleph_n^{\aleph_0} = \aleph_n$ for all $n < \omega$, i.e., CH, to obtain $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}$. Let F be a free group of rank $\aleph_\omega$ and G the cotorsion-hull of F. Then $|G| = |F|^{\aleph_0} = \aleph_{\omega+1}$. Now suppose $A \subseteq G$ is balanced, $|A| = \aleph_\omega$. Since A is pure in G, $\overline{A}$, the cotorsion-hull of A, is contained in G; i.e., $\overline{A}/A \subset G/A$. Now $\overline{A}/A$ is divisible, which implies $A = \overline{A}$ since A is balanced. Hence A is torsion-free cotorsion of size $\aleph_\omega$. In our setting, this is not possible for cardinality reasons.

In conclusion, there *are* torsion-free groups of cardinality $\aleph_{\omega+1}$ *without* any balanced subgroups of cardinality $\aleph_\omega$ (at least under CH). In our example above, A is still separable in G since $\overline{A}$ is a summand of G and $\overline{A}/A$ is divisible. At this time, we do not know of an example of such a G without large *separable* subgroups. An immediate consequence of Corollary 5.7 and [AH, 6.3] is the noteworthy

Theorem 5.8. *Assume CH holds. If G is a torsion-free group of cardinality $\leq \aleph_\omega$, then $\mathrm{Bext}^2(G, T) = 0$ for any torsion group T.*

Remark. For $|G| = \aleph_1$ this is contained in [AH, Theorem 6.3]. Using the long Hom-Bext sequence established in [Hu], we obtain the following corollary.

Corollary 5.9. *Assume CH holds. Then any balanced subgroup of a completely decomposable group of cardinality $\leq \aleph_\omega$ is a Butler group.*

6. SINGULAR COMPACTNESS

Here we state a version of Shelah's singular compactness analog to the version in [H] tailored to our purposes.

Definition 6.1. Let λ be a cardinal and B a group. A family $\mathscr{F}$ of subgroups of B is called a λ-family if

(a) $0 \in \mathscr{F}$.

(b) If $A \in \mathscr{F}$, then $|A| < \lambda$.

(c) If $\mu < \lambda$ and A_α, $\alpha < \mu$, is an ascending chain of subgroups in $\mathscr{F}$, then $\bigcup_{\alpha < \mu} A_\alpha \in \mathscr{F}$ if for some κ, $|A_\alpha| \leq \kappa < \lambda$ for all $\alpha < \mu$, $\aleph_0 < \mu$ regular.

(d) If $A \in \mathscr{F}$, and $X \subseteq B$ with $|X| < \lambda$, then there is $C \in \mathscr{F}$ with $|C| \leq |A||X|$ and $A \cup X \subseteq C$.

We adapt a proof in [H] to prove the next theorem. We give only an outline to indicate where there are changes.

Theorem 6.2 (Hodges/Shelah). *Let $\lambda \leq \aleph_\omega$ be a singular cardinal, B an abelian group of cardinality λ, and $\mathscr{F}$ a λ-family of subgroups of B. Let $\mathscr{F}^*$ be the family of unions of countable chains of $\mathscr{F}$.*

(j) For each $A \in \mathscr{F}^*$ there is an axiom 3-family $\mathscr{F}(A)$ of subgroups of A; cf. [AH].

(ij) If $A_n \subseteq A_{n+1} \in \mathscr{F}$ for all $n < \omega$ and if $A_n \in \mathscr{F}(A_{n+1})$ for each n, then there are axiom 3-families $\widetilde{\mathscr{F}}(A_n)$ of A_n and $\widetilde{\mathscr{F}}(A_n) \subseteq \mathscr{F}(A)$ where $A = \bigcup_{n < \omega} A_n$, provided $|A| < \lambda$.

(iij) If κ is a regular cardinal $< \lambda$ and $C_i \in \mathscr{F}$, $i < \kappa$, is a smooth chain with $|C_i| < \kappa$ for all $i < \kappa$, we set $C = \bigcup_{i < \kappa} C_i$ and define $\mathscr{F}(C) | C_i = \{X \mid X \in \mathscr{F}(C), X \subseteq C_i\}$. If $C_i \in \mathscr{F}(C)$, then $\mathscr{F}(C) | C_i$ is an axiom 3-subfamily of $\mathscr{F}(C_i)$. Note that $C \in \mathscr{F}$ since $|C_i| < \kappa < \lambda$ and κ regular.

Then there is a smooth chain A^i, $i < cf(\lambda)$, with $A^i \in \mathscr{F}^*$, $A^i \in \mathscr{F}(A^{i+1})$, and $B = \bigcup_{i < cf(\lambda)} A^i$.

Proof. (We adopt the proof in [H] up to some minor changes.) Let κ be a regular cardinal $< \lambda$. We define the κ-Shelah game on B: Player I picks subgroups B_i of cardinality $< \kappa$ for i even, $i = 0, 2, 4, \ldots$, and player II picks $B_i \in \mathscr{F}$ for i odd such that $B_i \subseteq B_{i+1}$ for all $i < \omega$. Player II wins if $B_i \in \mathscr{F}(B_{i+1})$ for all odd i. Observe that $\bigcup_{i < \omega} B_i \in \mathscr{F}^*$ since $B_{2_{j+1}} \in \mathscr{F}$ and $|B_i| < \kappa < \lambda$. Lemma 1.2 in [H] still holds: the κ-Shelah game on B is determinate; i.e., if player I has no winning strategy, then II has one! Thus, we have only to show that player I has no winning strategy. By way of contradiction, let us assume I has a winning strategy s. We have to show that II can defeat s. Let's play! Player I moves first and picks B_0. Now player II (he's got all the time in the world) gets working. By transfinite induction he constructs a

smooth chain C_i $(i < \kappa)$ of subgroups of B with $C_i \in \mathcal{F}^*$, $C_i \in \mathcal{F}$ if $(i) \neq \omega$, and $|C_i| < \kappa$ for all $i < \kappa$ such that: For each $j > \kappa$ and each finite set of ordinals $j_1 < \cdots < j_n < j$ each subgroup showing up in an initial piece (of length $2n$) of the κ-Shelah game where I plays his strategy s and starts with B_0 and II plays $C_{j_1}, \ldots, C_{j_n}$ is contained in C_j. This can be done since $j < \kappa$, κ regular and (d). Because (c), $C = \bigcup_{i<\kappa} C_i \in \mathcal{F}$. Since there is the axiom 3-family $\mathcal{F}(C)$ on C, we may assume—passing to a on $b \subseteq \kappa$—that $C_i \in \mathcal{F}(C)$ for all $i < \kappa$. Thus $C_i \in \mathcal{F}(C) \mid C_i \subseteq \mathcal{F}(C) \mid C_{i+1} \subseteq \mathcal{F}C(C_{i+1})$, i.e., $C_i \in \mathcal{F}(C_{i+1})$.

Now II easily defeats I by choosing B_{i+1} to be $C_{\alpha(i)}$ where $\alpha(i) = 1 + \inf\{\alpha < \kappa \mid B_0, B_1, \ldots, B_i \subseteq C_\alpha\}$. Now let (κ_i), $i < cf(\lambda)$, be a strictly increasing sequence of regular cardinals, $\sup\{\kappa_i \mid i < cf(\lambda)\} = \lambda$ and $\kappa_0 > cf(\lambda)$. Since $\mathcal{F}$ is a λ-family on B, we may write $B = \bigcup_{i<cf(\lambda)} C^i$, $|C^i| = \kappa_i$, $C^i \in \mathcal{F}$. We want to extend C^i to A^i, $i < cf(\lambda)$, smooth chain, $A^i \in \mathcal{F}^*$, and $A^i \in \mathcal{F}(A^{i+1})$ for all $i < cf(\lambda)$. Each A^i will be the union of a chain $C^i = C_0^i \subseteq B_0^i \subseteq A_0^i \subseteq C_1^i \subseteq B_1^i \subseteq A_1^i, \subseteq \cdots$ with

(i) If $i - 1$ exists, then $C_0^i, B_0^i, C_1^i \subseteq B_1^i, \ldots$ is a play of the κ_i^+-Shelah game on B where II uses his winning strategy picking the B_k^i's (I picks the C_k^i's). If i is a limit or 0, set $B_k^i = C_k^i$ for all $k < \omega$. We make player I (each upper index stands for a separable game) pick the C_k^i's. (As in [H], the A_n^i will be defined and used later to ensure that the chain $\{A^i\}$ is smooth.) We also make I pick the C_k^i's such that:

(ii) Whenever A_n^j is defined for each $j < cf(\lambda)$, then $C_{n+1}^i \supseteq \bigcup_{j \leq i} A_n^j$ and for $k = 0, 1, 2, \ldots, n$ $C_{n+1}^i \cap B_k^{i+1} \in \widetilde{\mathcal{F}}(B_k^{i+1})$ where $\widetilde{\mathcal{F}}(B_k^{i+1}) \subseteq \mathcal{F}(\bigcup_{k<\omega} B_k^i)$ is the family in hypothesis (ij). This can be done by using a standard back-and-forth argument using the fact that $\widetilde{\mathcal{F}}(B_k^{i+1})$ is closed with respect to (countable) unions.

Now define $A^i = \bigcup_{n<\omega} A_n^i = \bigcup_{n<\omega} C_n^i = \bigcup_{n<\omega} B_n^i$. We have $A^i = A^i \cap A^{i+1} = A^i \cap (\bigcup_{k<\omega} B_k^{i+1}) = \bigcup_{k<\omega} (A^i \cap B_k^{i+1}) = \bigcup_{k<\omega} ((\bigcup_{k<n<\omega} C_n^i) \cap B_k^{i+1}) = \bigcup_{k<\omega} (\bigcap_{k<n<\omega} (C_n^i \cap B_k^{i+1}))$. Since $C_n^i \cap B_k^{i+1} \in \widetilde{\mathcal{F}}(B_k^{i+1})$ for all $k \leq n$, we conclude $C_n^i \cap B_k^{i+1} \in \mathcal{F}(A^{i+1})$, $A^{i+1} = \bigcup_{k<\omega} B_k^{i+1}$, and since $\mathcal{F}(A^{i+1})$ is closed with respect to countable unions of subgroups of size κ_i, we conclude that $A^i \in \mathcal{F}(A^{i+1})$. Now we may complete the proof as in [H, p. 210] by defining the A_n^i's to ensure that $\{A^i \mid i < cf \lambda\}$ is smooth.

We are interested in

Corollary 6.3. *Let B be a Butler group of singular cardinality λ and assume that B admits to a λ-family $\mathcal{F}$ of B_2-subgroups. Then B is a B_2-group.*

Proof. For $A \in \mathcal{F}$ we let $\mathcal{F}(A)$ be an axiom 3-family of decent subgroups as introduced in [AH]. We have to show that 6.2 applies: (j) takes care of itself. To verify (ij), let $A_n \subset A_{n+1}$, $n < \omega$, be a chain in $\mathcal{F}$ and $A_n \in \mathcal{F}(A_{n+1})$. Thus

A_n is decent in (A_{n+1}) and—better—shows up in a B_2-filtration of A_{n+1}. Thus $A = \bigcup_{\alpha < \mu} H_\alpha = \bigcup_{n < \omega} A_n$, $H_{\alpha+1} = H_\alpha + G_\alpha$, G_α pure of finite rank and there is a sequence $\alpha_n < \alpha_{n+1} < \mu$ with $\mu = \sup\{\alpha_n \mid n < \omega\}$ and $H_{\alpha_n} = A_n$. Use this B_2-filtration of A and the A_n's to define "closed" subsets of μ and α_n as in [AH]. This gives rise to an axiom 3-family $\widetilde{\mathcal{F}}(A) \subseteq \mathcal{F}(A)$, $\widetilde{\mathcal{F}}(A) \mid A_n = \mathcal{F}(A_{n+1}) \subseteq \mathcal{F}(A_{n+1})$, and since $S \subseteq \alpha_n$ is "closed" iff S is closed in μ, $\mathcal{F}(A_n) \subseteq \widetilde{\mathcal{F}}(A) \subseteq \mathcal{F}(A)$. This shows that (ij) holds. Let the C_i's in (iij) play the role of the A_n and (iij) follows.

7. THE STRUCTURE OF BUTLER GROUPS

We will give a cardinal induction to show that under CH every Butler group of cardinality $\leq \aleph_\omega$ is a B_2-group. First we want to cover the "regular case."

Theorem 7.1. *Let G be a Butler group of regular cardinality κ such that G has a filtration $G = \bigcup_{\alpha < \kappa} G_\alpha$ and each G_α is a Butler group with $|G_\alpha| < \kappa$. Then there is a cub $C \subseteq \kappa$ and for any $\alpha \in C$, we have that G_α has the TEP in each G_β, $\beta \geq \alpha$.*

Proof. Let $E = \{\alpha \in \kappa \mid$ There is a $\alpha < \beta < \kappa$ such that G_α does not have TEP in $G_\beta\}$. By way of contradiction we assume that E is stationary in κ. We may assume that if $\alpha \in E$, then G_α does not have the TEP in $G_{\alpha+1}$. Thus there exist a (Σ-cyclic) torsion group T_α of cardinality $\leq |G_{\alpha+1}|$ and a homomorphism $\varphi_\alpha : G_\alpha \to T_\alpha$ that does not lift to $G_{\alpha+1}$. Now define $S_\alpha = \bigoplus_{\beta < \alpha} T_\beta$. (Here we borrow an idea from [EF].) By transfinite induction we construct a direct system $\{M_\alpha(\alpha < \kappa); \Pi_\alpha^\beta\}$ ([F1, p. 53 ff]) as follows: For any $\alpha < \beta < \kappa$ we have

(1) M_α is a mixed group with torsion subgroup S_α and $M_\alpha / S_\alpha \cong G_\alpha$.

(2) $0 \to S_\alpha \to M_\alpha \to G_\alpha \to 0$ is balanced exact.

(3) All Π_α^β are embeddings and $\Pi_\alpha^\beta \mid S_\alpha = \mathrm{id}$.

(4) If λ is a limit ordinal, then $M_\lambda = \varinjlim_\lambda M_\alpha$ and $\Pi_\alpha^\lambda : M_\alpha \to M_\lambda$, $\alpha < \lambda$, are the natural maps induced by Π_α^β, $\alpha < \beta < \lambda$.

(5) For all $\alpha < \beta$ we have $\Pi_\alpha^\beta(M_\alpha) + S_\beta = \Pi_\alpha^\beta(M_\alpha) \oplus S_{\beta,\alpha}$ where $S_{\beta,\alpha} = \bigoplus_{\alpha \leq r < \beta} T_r$.

Note that (5) implies that for any limit ordinal λ and $\alpha < \lambda$, $\Pi_\alpha^\lambda(M_\alpha) + S_\lambda = \Pi_\alpha^\lambda(M_\alpha) \oplus S_{\lambda,\alpha}$ and therefore S_λ is balanced in M_λ since $\Pi_\alpha^\lambda(S_\alpha)$ is balanced in $\Pi_\alpha^\lambda(M_\alpha)$. (Keep in mind that all Π_α^λ are embedding.)

(6) If M_α is already defined and $\alpha \in E$, then $M_\alpha = S_\alpha \oplus B_\alpha$ and $B_\alpha \cong G_\alpha$. Let $\mu_\alpha : G_\alpha \to B_\alpha$ be an isomorphism. This isomorphism extends to an isomorphism $\mu_{\alpha+1} : G_{\alpha+1} \to B_{\alpha+1}$ where $B_{\alpha+1} \supseteq B_\alpha$. We set $M_{\alpha+1} = S_{\alpha+1} \oplus B_{\alpha+1}$ and define $\Pi_\alpha^{\alpha+1}(s, b) = (s + \varphi_\alpha(\mu_\alpha^{-1}(b)), b)$ for all $s \in S_\alpha$, $b \in B_\alpha$. Clearly, $\Pi_\alpha^{\alpha+1} : M_\alpha \to M_{\alpha+1}$ is an embedding.

If $\alpha \notin E$, we define $\Pi_\alpha^{\alpha+1}$ as above with $\varphi_\alpha = 0$. We have to show that (5)

is satisfied. Let $\beta \leq \alpha$. Then

$$S_{\alpha+1} + \Pi_\beta^{\alpha+1}(M_\beta) = \Pi_\alpha^{\alpha+1}(\Pi_\beta^\alpha(M_\beta)) + S_{\alpha+1} \subseteq M_\alpha + S_{\alpha+1}$$
$$= M_\alpha + S_{\alpha+1,\alpha} = (M_\alpha + S_\alpha) \oplus S_{\alpha+1,\alpha}.$$

By induction hypothesis, $\Pi_\beta^\alpha(M_\beta) + S_\alpha = \Pi_\beta^\alpha(M_\beta) + S_{\alpha,\beta}$. Thus $S_{\alpha+1} + \Pi_\beta^{\alpha+1}(M_\beta)$
$= \Pi_\beta^\alpha(M_\beta) \oplus S_{\alpha,\beta} \oplus S_{\alpha+1,\alpha} = \Pi_\beta^\alpha(M_\beta) \oplus S_{\alpha+1,\beta}$. If λ is a limit and $\beta < \lambda$,
then

$$\Pi_\beta^\lambda(M_\beta) + S_\lambda = \Pi_\beta^\lambda(M_\beta) + S_{\lambda,\beta} = \bigcup_{\beta \leq \alpha < \lambda} (\Pi_\beta^\lambda(M_\beta) \oplus S_{\alpha,\beta})$$
$$= \Pi_\beta^\lambda(M_\beta) \oplus \bigcup_{\beta \leq \alpha < \lambda} S_{\alpha,\beta} = \Pi_\beta^\lambda(M_\beta) \oplus S_{\lambda,\beta}.$$

Let $M_k = \varinjlim_\kappa M_\alpha$. Then $M_\kappa = \bigcup_{\alpha < \kappa} \Pi_\alpha^k(M_\alpha)$ and $\Pi_\alpha^\kappa(M_\alpha) + S_\kappa = \Pi_\alpha^\kappa(M_\alpha) \oplus S_{\kappa,\alpha}$. Therefore $S_\kappa = \bigoplus_{\alpha < \kappa} T_\alpha$ is balanced in M_κ with $M_\kappa/S_\kappa \cong G$. Since G is a Butler group, there is a splitting map $\psi : G \to M_\kappa$. Since κ is regular and $|M_\alpha| < \kappa$ for all α, there is a cub $C \subseteq \kappa$ such that $\psi(G_\alpha) \subseteq \Pi_\alpha^\lambda(M_\alpha)$ for all $\alpha \in C$. Since E is stationary, there is some $\alpha \in E \cap C$. Let $\beta \in C$ with $\beta > \alpha$. Then $\psi \mid G_\beta : G_\beta \to \Pi_\beta^\lambda(M_\beta)$ is a splitting map. We may apply $(\Pi_\beta^\lambda)^{-1}$ and we may assume that $\psi : G_\beta \to M_\beta$ and $\psi \mid G_\alpha \to \Pi_\alpha^\beta(M_\alpha)$ is a splitting map. We have $\psi(g) = (h(g), \mu_\beta(g))$ for all $g \in G_\beta$ and $h = h_0 \oplus h_1 : G_\beta \to S_{\alpha+1} \oplus S_{\beta,\alpha+1}$, $h_1 \mid G_\alpha = 0$. If $g \in G_\alpha$, then $h(g) \in S_\alpha$. We may subtract h_1 from ψ and thus are able to assure $h_1 = 0$. Then $\psi : G_{\alpha+1} \to M_{\alpha+1}$ is a splitting map and

$$\Pi_\alpha^{\alpha+1}(M_\alpha) = \{s + \varphi_\alpha(\mu_\alpha^{-1}(b)), b) \mid b \in B_\alpha, s \in S_\alpha\} \subseteq S_\alpha$$
$$\oplus T_\alpha \oplus \{(h(\mu_\alpha^{-1}(b)), b) \mid b \in B_\alpha\}$$

and $\Pi_\alpha^{\alpha+1}(M_\alpha) = S_\alpha \oplus \{h(\mu_\alpha^{-1}(b), b) \mid b \in B_{\alpha+1}\}$. Thus $\varphi_\alpha(\mu_\alpha^{-1}(b)), b) \equiv h(\mu_\alpha^{-1}(b)) \mod S_\alpha$ for all $b \in B_\alpha$. Let $\Pi : S_{\alpha+1} \to T_\alpha$ be the projection. Then $\Pi \circ \varphi_\alpha = \varphi_\alpha$ and $\varphi_\alpha(\mu_\alpha^{-1}(b)) = (\Pi \circ h)(\mu_\alpha^{-1}(b))$ for all $b \in B_\alpha$. This implies $\varphi_\alpha = (\Pi \circ h) \mid G_\alpha$ and φ_α lifts to $G_{\alpha+1}$, a contradiction. This shows that E is not stationary and therefore we may assume that $G = \bigcup_{\alpha < \kappa} G_\alpha$ is a filtration where each G_α has TEP in $G_{\alpha+1}$.

Next we will modify Lemma 3.1 in [AH]. We refer to [AH] for a definition of Hill's compatibility relation $\|$.

Lemma 7.2. *Let A, B, and H be pure subgroups of the torsion-free group G.*
   (a) *If A is a balanced in G and $A \| H$, then $A \cap H$, is balanced in H.*
   (b) *If $A \| H$ and $H + A \| B$ then $H \| A + B$.*

Proof. To prove (a), let $h \in H - (H \cap A) = H - A$. For any $a \in H \cap A \subseteq A$ we have $|h + a| \leq |h + a_0|$ where $a_0 = a_0(h) \in A$ since A is balanced in G. There is some element $a_1 \in A \cap H$ such that $|h + a| \leq |h + a_0| \leq |h + a_1|$ for all $a \in A \cap H$. Thus $A \cap H$ is balanced in H. To show (b), fix some

$h \in H$, $a \in A$, $b \in B$. Then $|h + a + b| = |(h + a) + b| \leq |h + a + b_0|$ where $b_0 \in (H + A) \cap B$, i.e., $b_0 = h_0 + a_0 \in B$ and $h_0 \in H$, $a_0 \in A$. Thus $|h + a + b| \leq |h + a + h_0 + a_0| \leq |(h + h_0) + (a + a_0)| \leq |h + h_0 + x|$ for some $x \in A \cap H$ since $H \parallel A$. This implies $|h + a + b| \leq |h + (b_0 - a_0) + x|$ and since $a_0, x \in A$, $b_0 \in B$ we have $y = b_0 - a_0 + x = h_0 + x \in (A + B) \cap H$. This shows $H \parallel A + B$.

Definition 7.3 [FH]. Let G be a group, λ a cardinal, and $\mathcal{F}$ a family of subgroups of G. We call $\mathcal{F}$ a $G(\lambda)$-family if

  (a) $0 \in \mathcal{F}$.
  (b) If $\{F_i \mid i \in I\} \subseteq \mathcal{F}$ is a chain then $\bigcup_{i \in I} F_i \in \mathcal{F}$.
  (c) If $F \in \mathcal{F}$ and $A \subseteq G$, $|A| \leq \lambda$, then there is $F' \in \mathcal{F}$ with $A + F/F \subseteq F'/F$ and $|F'/F| \leq \lambda$.

Definition 7.4. Let $C = \bigoplus_{i \in I} R_i$ be a completely decomposable group. For any subset $J \subseteq I$ we set $C(J) = \bigoplus_{i \in J} R_i$. If H is a subgroup of C, we call a subset $J \subseteq I$ an H-special set if $C(J) \parallel H$. Moreover, we define $H(J) = H \cap C(J)$.

Theorem 7.5. *Let* H *be a pure subgroup of the completely decomposable group* $C = \bigoplus_{i \in I} R_i$. *Then* H *admits a* $G(2^{\aleph_0})$-*family* $\mathcal{F}_H$ *of balanced subgroups induced by a* $G(2^{\aleph_0})$-*family* $\mathcal{F}_C$ *of direct summands of* C, *i.e.,* $\mathcal{F}_H = \{X \cap H \mid X \in \mathcal{F}_C\}$.

Proof. Let $\mathcal{F}_H$ be the family of H-special subsets of I and $\mathcal{F}_H^{\#} = \{C(X) \mid X \in \mathcal{F}_H\}$. We want to show that $\mathcal{F}_H^{\#}$ is a $G(2^{\aleph_0})$ family of C. We have only to show 7.3(c). So let $C(J) \in \mathcal{F}_h^{\#}$ and $A \subseteq C$, $|A| \leq 2^{\aleph_0}$. By induction on $n < \omega$ we define subsets $Z_n \subseteq I$, $|Z_n| \leq 2^{\aleph_0}$: Let Z_0 be minimal with $A \subseteq C(Z_0)$. Suppose Z_n is already defined. For $g \in C(Z_n)$ we find a set $\{h_{\alpha, g} \mid \alpha < 2^{\aleph_0}) \subseteq H + C(J)$ such that for any $y \in H + C(J)$ we have $|y + g| \leq |h_{\alpha, g} + g|$ for some $\alpha < 2^{\aleph_0}$. (Note that there are only $2^{\aleph_0}$ many height sequences.) Now let Z_{n+1} be minimal with $C(Z_n) \cup \{h_{\alpha, g} \mid g \in C(Z_n), \alpha < 2^{\aleph_0}\} \subseteq C(Z_{n+1})$. Note that $|Z_{n+1}| \leq 2^{\aleph_0}$. Let $Z = \bigcup_{n < \omega} Z_n$. Then $H + C(J) \parallel C(Z)$ and $H \parallel C(J)$ since $J \in \mathcal{F}_H$. By 7.2(b), $H \parallel C(J) + C(Z) = C(J \cup Z)$ and by 7.2(a), $H(J^{\#}) = H \cap C(J^{\#})$, $J^{\#} = J \cup Z$ is a balanced subgroup of H. Now we are done since $|J^{\#} - J| \leq |Z| \leq 2^{\aleph_0}$.

Corollary 7.6 (CH). *Let* H *be a balanced subgroup of a completely decomposable group such that* H *is a Butler group. Then* H *admits a* $G(\aleph_1)$-*family of Butler groups.*

Proof. Observe that a countable extension of a balanced subgroup of H is separable in H and Corollary 4.6.

   We are now ready to prove our main theorem.

Main Theorem (7.7). *Assume CH holds. Then any Butler group of cardinality* $\leq \aleph_\omega$ *is a* B_2*-group.*

Proof. By cardinal induction. Let G be a Butler group of cardinality κ and assume that each Butler group of cardinality $< \kappa \leq \aleph_\omega$ is a B_2-group. Note that $\kappa = \aleph_1$ has this property. Let $0 \to H \to C \to G \to 0$ be a balanced resolution of G; i.e., C is completely decomposable and H is balanced in C. We may assume that $|H| = |C| = \kappa$. By Theorem 7.5 there is a $G(\aleph_1)$-family $\mathscr{F}_c$ of summands of C such that $\mathscr{F}_H = \{H \cap X \mid X \in \mathscr{F}_c\}$ is a $G(\aleph_1)$-family of balanced subgroups of H. Therefore we may—for any $S \in \mathscr{F}_H$—write $H = \bigcup_{\alpha < \kappa} H_\alpha$ with $H_\alpha \in \mathscr{F}_H$ and $|H_{\alpha+1}/H_\alpha| \leq \aleph_1$ and $H_0 = S$. Since $|G| \leq \aleph_\omega$, Theorem 5.8 implies that $\mathrm{Bext}^2(G, T) = 0$. Since H is balanced in C, the long Hom-Bext sequence implies that $\mathrm{Bext}(H, T) = 0$; i.e., H is a Butler group. By Corollary 4.6 we obtain that each H_α is also a Butler group; i.e., any $S \in \mathscr{F}_H$ is a Butler group and by induction hypothesis, H has a $G(\aleph_1)$-family $\mathscr{F}_H$ of B_2-groups.

If $\kappa \leq \aleph_\omega$ is singular, Corollary 6.3 implies that H is a B_2-group. If κ is regular we may apply Theorem 7.1 and there is a filtration $H = \bigcup_{\alpha < \kappa} H_\alpha$ with $H_\alpha \in \mathscr{F}_H$, $|H_\alpha| < \kappa$, and each H_α has TEP in $H_{\alpha+1}$ Since $H_{\alpha+1}$ is a pure subgroup of C, each pure finite-rank subgroup of $H_{\alpha+1}$ is a Butler group of finite rank. Moreover, $H_{\alpha+1}/H_\alpha$ is a Butler group of cardinality $< \kappa$ by Corollary 4.3. Thus we may apply Proposition 3.9 and continue any B-filtration of H_α to one of $H_{\alpha+1}$. This and an obvious induction over $\alpha < \kappa$ show that H is a B_2-group. By Corollary 3.5 we may assume that each $A \in \mathscr{F}_H$ has the TEP in H. Recall $A = H \cap X$, $X \in \mathscr{F}_c$. If $X \in \mathscr{F}_c$ such that $X \cap H$ has TEP in H, then $X \cap H$ has TEP in C since H does. Therefore $X/X \cap H \cong X + H/H$ is a Butler group and we may assume that G admits a $G(\aleph_1)$-family $\mathscr{F}_G$ of Butler groups, namely $\mathscr{F}_G = \{X + H/H \mid X \in \mathscr{F}_c, |X| < \kappa\}$. By induction hypothesis, all groups in $\mathscr{F}_G$ are B_2-groups. If κ is singular, G is a B_2-group by Corollary 6.3. Thus we may assume that κ is regular. Again, Theorem 7.1 and properties of $\mathscr{F}_G$ show the existence of a filtration $G = \bigcup_{\alpha < \kappa} G_\alpha$ into B_2-groups G_α, $|G_\alpha| < \kappa$ and each G_α has the TEP in $G_{\alpha+1}$. By Theorem 3.7(i), each pure finite-rank subgroup of G is Butler since H is a B_2-group. In the same way as above for H, we utilize 4.3 and 3.7 to show that G is a B_2-group.

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